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SCIENCE DIRECT
ELSEVIER Computers and Mathematics with Applications 52 (2006) 539–554
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Limit-Point and Limit-Circle Criteria for Singular Second-Order Linear Difference Equations with Complex Coefficients

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(Received April 2005; accepted September 2005)

Abstract—This paper is concerned with limit-point and limit-circle criteria of singular second-order linear difference equations with complex coefficients. Formally self-adjoint second-order linear difference operators are formulated. Several sufficient conditions and sufficient and necessary conditions for the limit-point and limit-circle cases are established. These results extend some relevant existing results. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Second-order linear difference equation, Complex coefficient, Limit-point case, Limit-circle case.

1. INTRODUCTION

Consider the following second-order linear difference equation:

$$\begin{aligned} ly(n) := w^{-1}(n)\,\{-\nabla(p(n)\Delta y(n)) + q(n)y(n) \\ +i[\Delta(r(n)y(n)) + r(n)\nabla y(n)]\} = \lambda y(n), \qquad n \in [0,\infty), \end{aligned} \tag{1.1}$$

where ∇ and Δ are the backward and forward difference operators, respectively, namely, $\nabla y(n) := y(n) - y(n-1)$ and $\Delta y(n) := y(n+1) - y(n)$; $p(n)$, $q(n)$, $r(n)$, and $w(n)$ are real numbers with $w(n) > 0$ for $n \in [0,\infty)$ and $p(n) \neq 0$ for $n \in [-1,\infty)$; $[0,\infty)$ denotes the integer set $\{n\}_{n=0}^{\infty}$; λ is a complex parameter.

If $r(n) \equiv 0$, (1.1) becomes the following formally self-adjoint second-order difference equation with only real coefficients:

$$ly(n) := w^{-1}(n)\{-\nabla(p(n)\Delta y(n)) + q(n)y(n)\} = \lambda y(n), \qquad n \in [0,\infty), \tag{1.2}$$

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This research was supported by the NNSF of China (Grant 10471077) and the Shandong Research Funds for Young Scientists (Grant 03BS094).

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doi:10.1016/j.camwa.2006.08.024

which can be regarded as a discrete analog of the following self-adjoint second-order differential equation:

$$\tilde{l}y(t) := w^{-1}(t)\{-(p(t)y'(t))' + q(t)y(t)\} = \lambda y(t), \qquad t \ge 0. \tag{1.3}$$

Weyl [1] first made an important observation that $\tilde{l}$ can be divided into two cases: the limit-point and limit-circle cases. His work has greatly been developed and generalized to Hamiltonian systems (cf., [2–8], and references cited therein). Atkinson first found that the formally self-adjoint second-order difference equation (1.2) can also be divided into the limit-point case (l.p.c.) and the limit-circle case (l.c.c.) and established several criteria of the limit-point and limit-circle cases for equation (1.2) [5]. His work was further developed (cf., [9–13], and references cited therein). Several other sufficient conditions for the limit-point and limit-circle cases of equation (1.2) were given later (cf., [9–11,13]). Especially, some sufficient and necessary conditions were given in the recent paper [13]. Research on spectral theory of discrete Hamiltonian systems has attracted a great deal of interest and some good results have been obtained (cf., [14–17], and references cited therein). In particular, some Titchmarsh-Weyl fundamental theory has been established in the paper [17] for the following singular discrete Hamiltonian system:

$$J\Delta y(t) = (\lambda W(t) + P(t))\, R(y)(t), \qquad t \in [0, \infty), \tag{1.4}$$

where $W(t)$ and $P(t)$ are $2d \times 2d$ Hermitian matrices, and $W(t) \ge 0$ is the weighted function; the right partial shift operator $R(y)(t) := (y_1^{\top}(t+1), y_2^{\top}(t))^{\top}$ with $y(t) = (y_1^{\top}(t), y_2^{\top}(t))^{\top}$, and $y_1(t), y_2(t) \in \mathbf{C}^d$;

$$J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix},$$

and I_d is the $d \times d$ unit matrix. For convenience, in the sequel discussions, we first recall the assumptions for system (1.4) in [17]. It is assumed that $W(t)$ has the block diagonal form $W(t) = \operatorname{diag}\{W_1(t), W_2(t)\}$, where $W_j(t)$ is a $d \times d$ nonnegative matrix for $j = 1, 2$. $P(t)$ can be written as the block form:

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix},$$

where $A(t)$, $B(t)$, and $C(t)$ are $d \times d$ matrices and $A^*(t)$ is the conjugate transpose of $A(t)$. Furthermore, it is assumed that

(A_1) there exists a positive integer n_0 such that for all $\lambda \in \mathbf{C}$ and for all the nontrivial solution $y(t,\lambda)$ of (1.4), it holds that

$$\sum_{t=0}^{k} R(y)^*(t,\lambda) W(t) R(y)(t,\lambda) > 0, \qquad k \ge n_0;$$

(A_2) $I_d - A(t)$ is invertible on $[0, \infty)$.

A classification for the discrete Hamiltonian system (1.4) was given in terms of the defect indices of the corresponding minimal operator [17, Definition 5.1]. In addition, several equivalent conditions to the limit-point and limit-circle cases for system (1.4) were established in [17].

It has been noticed that many criteria of the limit-point and limit-circle cases were established only for equation (1.2), whose coefficients are all real. However, there are few criteria of the limit-point and limit-circle cases for equation (1.1) in terms of its coefficients. This paper focuses on studying equation (1.1) and establishing several criteria of the limit-point and limit-circle cases for equation (1.1), some of which extend relevant existing results for equation (1.2).

By setting $v(n) = p(n)\Delta y(n) - ir(n+1)y(n+1)$, (1.1) can be rewritten as the following discrete Hamiltonian system:

$$J\Delta x(n) = \left(\lambda \hat{W}(n) + \hat{P}(n)\right) R(x)(n), \qquad n \in [-1, \infty), \tag{1.5}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{P}(n) = \begin{pmatrix} p^{-1}(n)r^2(n+1) - q(n+1) & -ip^{-1}(n)r(n+1) \\ ip^{-1}(n)r(n+1) & p^{-1}(n) \end{pmatrix},$$

$\hat{W}(n) = \operatorname{diag}\{w(n+1), 0\}$, and $R(x)(n) = (y(n+1), v(n))^{\mathrm{T}}$ with $x(n) = (y(n), v(n))^{\mathrm{T}}$. It is easy to verify that (A_1) and (A_2) hold in this case. This paper will apply the classification for discrete Hamiltonian systems and some results of [17].

The rest of the paper is organized as follows. In Section 2, a formally self-adjoint second-order difference expression is first formulated and some fundamental properties of solutions of equation (1.1) are studied. Section 3 is devoted to criteria of the limit-point case for equation (1.1). Two sufficient conditions and a sufficient and necessary condition subject to a certain restriction are obtained. In Section 4, several sufficient conditions and a sufficient and necessary condition of the limit-circle case for equation (1.1) are established.

2. SET UP

Introduce the following space:

$$l_w^2(0,\infty) := \left\{ y : y = \{y(n)\}_{n=-1}^{\infty} \subset \mathbf{C} \text{ and } \sum_{n=0}^{\infty} w(n)|y(n)|^2 < \infty \right\}.$$

Obviously, $l_w^2(0,\infty)$ is a Hilbert space with the inner product $\langle x, y \rangle := \sum_{n=0}^{\infty} w(n)\bar{x}(n)y(n)$ for any x, $y \in l_w^2(0,\infty)$, where $\bar{x}$ is the complex conjugate of x. Denote $\|y\|_w := (\langle y, y \rangle)^{1/2}$ for $y \in l_w^2(0,\infty)$. When $w(n) \equiv 1$ for $n \geq 0$, $l_w^2(0,\infty)$ and $\|y\|_w$ are denoted by $l^2(0,\infty)$ and $\|y\|$, respectively. For convenience, y is called to be square summable if $y \in l_w^2(0,\infty)$.

For formally self-adjoint higher-order differential expressions, we refer to [18]. The formally self-adjoint second-order difference expression in (1.2) with only real coefficients was given in [5,11]. Now, we formulate formally self-adjoint second-order difference expressions with complex coefficients.

The general second-order linear difference operator can be written as

$$My(n) := a(n)\nabla\Delta y(n) + b(n)\Delta y(n) + c(n)y(n), \qquad n \in [n_1, n_2], \tag{2.1}$$

where $a(n)$, $b(n)$, and $c(n)$ are complex numbers; n_1 and n_2 are finite integers or $n_1 = -\infty$, $n_2 = +\infty$; and $[n_1, n_2]$ denotes the integer set $\{n\}_{n=n_1}^{n_2}$. For any y, $z \in l[n_1 - 1, n_2 + 1] := \{y : y = \{y(n)\}_{n_1-1}^{n_2+1} \subset \mathbf{C}\}$, we have

$$\begin{aligned} \bar{z}(n)My(n) = {} & [\nabla\Delta(a(n)\bar{z}(n)) - \nabla(b(n)\bar{z}(n)) + c(n)\bar{z}(n)]\, y(n) \\ & + \nabla\left[\bar{z}(n)a(n)\Delta y(n) - \Delta(\bar{z}(n)a(n))\, y(n) + \bar{z}(n)b(n)y(n+1)\right]. \end{aligned}$$

So, it is natural to give the following definitions.

DEFINITION 2.1. *The operator defined by*

$$M^{+}y(n) := \nabla\Delta(\bar{a}(n)\,y(n)) - \nabla\left(\bar{b}(n)\,y(n)\right) + \bar{c}(n)\,y(n) \tag{2.2}$$

is said to be the formally adjoint operator of M.

DEFINITION 2.2. *M is said to be formally self-adjoint if $M = M^{+}$.*

THEOREM 2.1. *M is formally self-adjoint if and only if M can be written as*

$$My(n) = -\nabla(p(n)\Delta y(n)) + q(n)y(n) + i\left[\Delta(r(n)y(n)) + r(n)\nabla y(n)\right], \qquad n \in [n_1, n_2], \quad (2.3)$$

where $p(n)$, $q(n)$, and $r(n)$ are real numbers.

PROOF. Suppose that M is formally self-adjoint. Let

$$a(n) = a_1(n) + ia_2(n), \qquad b(n) = b_1(n) + ib_2(n), \qquad c(n) = c_1(n) + ic_2(n),$$

where a_j, b_j, and c_j $(j = 1, 2)$ are real numbers. Then M and M^+ can be rewritten as

$$\begin{aligned} My(n) &= a_1(n)\nabla\Delta y(n) + b_1(n)\Delta y(n) + c_1(n)y(n) \\ &\quad + i[a_2(n)\nabla\Delta y(n) + b_2(n)\Delta y(n) + c_2(n)y(n)], \end{aligned} \tag{2.4}$$

$$\begin{aligned} M^+y(n) &= \nabla\Delta(a_1(n)y(n)) - \nabla(b_1(n)y(n)) + c_1(n)y(n) \\ &\quad - i[\nabla\Delta(a_2(n)y(n)) - \nabla(b_2(n)y(n)) + c_2(n)y(n)]. \end{aligned} \tag{2.5}$$

From (2.4), (2.5), and $M = M^+$, it follows that

$$\begin{aligned} \Delta a_1(n) &= b_1(n), \\ a_2(n+1) + a_2(n) + b_2(n) &= 0, \\ 2a_2(n) + b_2(n) &= c_2(n). \end{aligned} \tag{2.6}$$

Inserting (2.6) into (2.4), we have

$$My(n) = \nabla[a_1(n+1)\Delta y(n)] + c_1(n)y(n) - i[\Delta(a_2(n)y(n)) + a_2(n)\nabla y(n)],$$

which implies (2.3) by setting $p(n) = -a_1(n+1)$, $q(n) = c_1(n)$, and $r(n) = -a_2(n)$.

Conversely, suppose that M is expressed as (2.3). It can be easy to verify that $M = M^+$. Hence, M is formally self-adjoint. This completes the proof.

Now, we give the other form of the formally self-adjoint difference expression. This form will be useful in the rest discussions of the paper.

COROLLARY 2.1. *M is formally self-adjoint if and only if M can be written as*

$$My(n) = -c(n)y(n+1) + b(n)y(n) - \bar{c}(n-1)y(n-1), \qquad n \in [n_1, n_2], \qquad (2.3^*)$$

where $b(n)$ is a real number for $n \in [n_1, n_2]$.

PROOF. By Theorem 2.1, M is formally self-adjoint if and only if M can be written as the form (2.3). By setting $c(n) = p(n) - ir(n+1)$ and $b(n) = q(n) + p(n) + p(n-1)$ or by setting $p(n) = \operatorname{Re}\{c(n)\}$, $r(n) = -\operatorname{Im}\{c(n-1)\}$, and $q(n) = b(n) - \operatorname{Re}\{c(n)\} - \operatorname{Re}\{c(n-1)\}$, it can be easily concluded that the forms (2.3) and (2.3*) are equivalent, where $\operatorname{Re}\{z\}$ and $\operatorname{Im}\{z\}$ are the real and imaginary parts of the complex number z, respectively. Hence, the proof is complete.

REMARK 2.1. Equation (1.1) can be rewritten as

$$-c(n)y(n+1) + b(n)y(n) - \bar{c}(n-1)y(n-1) = \lambda w(n)y(n), \qquad n \in [0, \infty), \qquad (1.1^*)$$

where $b(n) = q(n) + p(n) + p(n-1)$ is a real number for $n \in [0, \infty)$ and $c(n) = p(n) - ir(n+1)$ is a complex number with $\operatorname{Re}\{c(n)\} \neq 0$ for $n \in [-1, \infty)$.

Based on the discussion in Section 1, equation (1.1) can be transformed into the Hamiltonian system (1.5). The natural difference operator corresponding to system (1.5) is

$$(\mathcal{L}x)(n) := J\Delta x(n) - \hat{P}(n)R(x)(n).$$

Let H_0 be the minimal operator induced by $\mathcal{L}$ (see [17, Section 2] for its definition) and $d_{\pm}$ be the positive and negative indices of H_0 [17, Section 2]. According to the classification for the discrete Hamiltonian system (1.4) [17, Definition 5.1], $\mathcal{L}$ is said to be in l.p.c. at $n = \infty$ if $d_+ = d_- = 1$, and $\mathcal{L}$ is said to be in l.c.c. at $n = \infty$ if $d_+ = d_- = 2$. On the other hand, it is evident that x is a solution of system (1.5) if and only if the first component y of x is a solution of equation (1.1) and they satisfy

$$\sum_{n=-1}^{\infty} R(x)^*(n)\hat{W}(n)R(x)(n) = \sum_{n=0}^{\infty} \bar{y}(n)w(n)y(n).$$

This implies that x is a solution in $l^2_{\hat{W}}(-1,\infty)$ of system (1.5) if and only if y is a solution in $l^2_w(0,\infty)$ of equation (1.1). Hence, the following result can be easily concluded by using the relation between the number of linear independent solutions in $l^2_{\hat{W}}(-1,\infty)$ of system (1.5) and the defect index [17, Theorem 5.1], and the largest defect index theorem [17, Theorem 5.5].

THEOREM 2.2. *If all the solutions of $ly = \lambda_0 y$ are in $l^2_w(0,\infty)$ for some complex number $\lambda_0 \in \mathbf{C}$, then all the solutions of $ly = \lambda y$ are in $l^2_w(0,\infty)$ for any $\lambda \in \mathbf{C}$. Consequently, $d_+ = d_- = 2$ in this case. Otherwise, $d_+ = d_- = 1$.*

By Theorem 2.2, $\mathcal{L}$ can be divided into two cases: the limit-point and limit-circle cases at $n = \infty$. Now we can give the following definitions of the limit-point and limit-circle cases for the operator l.

DEFINITION 2.3. *l is said to be in l.p.c. at $n = \infty$ if $\mathcal{L}$ is in l.p.c. at $n = \infty$; l is said to be in l.c.c. at $n = \infty$ if $\mathcal{L}$ is in l.c.c. at $n = \infty$.*

The following is a consequence of Theorem 2.2.

THEOREM 2.3. *l is in l.c.c. at $n = \infty$ if for some $\lambda_0 \in \mathbf{C}$, every solution of $ly = \lambda_0 y$ is in $l^2_w(0,\infty)$. Otherwise, l is in l.p.c. at $n = \infty$.*

REMARK 2.2. Usually, equation (1.1) is called to be in l.c.c. at $n = \infty$ if l is in l.c.c. at $n = \infty$; equation (1.1) is called to be in l.p.c. at $n = \infty$ if l is in l.p.c. at $n = \infty$.

LEMMA 2.1. *Let y and z be any two solutions of (1.1). Then, their Wronskian*

$$\begin{aligned} W[y,z](n) :&= \begin{vmatrix} y(n+1) & z(n+1) \\ (-p(n)+ir(n+1))\Delta y(n) & (-p(n)+ir(n+1))\Delta z(n) \end{vmatrix} \\ &= (p(n)-ir(n+1))\,(y(n+1)z(n)-y(n)z(n+1)) \end{aligned}$$

satisfies the following identity:

$$|W[y,z](n)| = |W[y,z](-1)|, \qquad \forall n \in [-1,\infty).$$

PROOF. Let y and z be any two solutions of (1.1). Since (1.1) can be written as the discrete Hamiltonian system (1.5), the Wronskian of y and z can be expressed as

$$W[y,z](n) = -\begin{vmatrix} y(n) & z(n) \\ v_y(n) & v_z(n) \end{vmatrix},$$

where $v_y(n) = p(n)\Delta y(n) - ir(n+1)y(n+1)$ and $v_z(n) = p(n)\Delta z(n) - ir(n+1)z(n+1)$. Then, this lemma directly follows from [17, Theorem 2.2]. The proof is complete.

The following result is easily concluded by Lemma 2.1 and by using uniqueness of solutions of initial value problems for equation (1.1).

PROPOSITION 2.1. *Let y and z be two solutions of (1.1). Then, y and z are linearly independent on $[-1,\infty)$ if and only if $W[y,z](-1) \neq 0$.*

For any fixed $\lambda \in \mathbf{C}$, let ϕ and θ be solutions of (1.1) satisfying the following initial conditions:

$$\phi(-1) = (p(-1) - ir(0))^{-1}, \qquad \phi(0) = \theta(-1) = 0, \quad \theta(0) = 1, \tag{2.7}$$

and let φ and ψ be solutions of (1.1*) satisfying the following initial conditions:

$$\varphi(-1) = \left(\operatorname{Re}\{c(-1)\} + i\operatorname{Im}\{c(-1)\}\right)^{-1}, \qquad \varphi(0) = \psi(-1) = 0, \quad \psi(0) = 1. \tag{2.8}$$

The following proposition is a direct consequence of Lemma 2.1 and Proposition 2.1.

PROPOSITION 2.2. *ϕ and θ are linearly independent solutions of (1.1) and satisfy*

$$|W[\phi,\theta](n)| = 1, \qquad n \in [-1,\infty), \tag{2.9}$$

and φ and ψ are linearly independent solutions of (1.1) and satisfy*

$$|W[\varphi,\psi](n)| = 1, \qquad n \in [-1,\infty). \tag{2.10}$$

THEOREM 2.4. *For any sequence $\{f(n)\}_{n=0}^{\infty} \subset \mathbf{C}$ and for any constants $c_1, c_2 \in \mathbf{C}$, the initial value problem*

$$ly(n) - \lambda y(n) = f(n), \qquad n \in [0,\infty), \tag{2.11}$$

$$y(-1) = c_1, \qquad y(0) = c_2, \tag{2.12}$$

has a unique solution y, which can be expressed as

$$y(n) = c_1\phi^{-1}(-1)\phi(n) + c_2\theta(n) - \sum_{j=0}^{n-1} \frac{w(j)\left(\phi(n)\theta(j) - \phi(j)\theta(n)\right) f(j)}{W[\phi,\theta](j)}, \tag{2.13}$$

for all $n \in [-1,\infty)$, where $\sum_{j=0}^{-2} \cdot = \sum_{j=0}^{-1} \cdot := 0$.

PROOF. First, show

$$\chi(n) = -\sum_{j=0}^{n-1} \frac{w(j)\left(\phi(n)\theta(j) - \phi(j)\theta(n)\right) f(j)}{W[\phi,\theta](j)}, \qquad n \in [-1,\infty), \tag{2.14}$$

is a solution of equation (2.11). Since $\chi(-1) = \chi(0) = 0$, it is easy to verify that χ satisfies equation (2.11) at $n = 0$. For any $n \in [1,\infty)$, it follows from (2.14) that

$$\begin{aligned} -\nabla(p(n)\Delta\chi(n)) &= \nabla(p(n)\Delta\phi(n)) \sum_{j=0}^{n-1} \frac{w(j)\theta(j)f(j)}{W[\phi,\theta](j)} - \nabla(p(n)\Delta\theta(n)) \sum_{j=0}^{n-1} \frac{w(j)\phi(j)f(j)}{W[\phi,\theta](j)} \\ &\quad + \frac{p(n)\left(\phi(n+1)\theta(n) - \phi(n)\theta(n+1)\right) w(n)f(n)}{W[\phi,\theta](n)}, \end{aligned}$$

$$\begin{aligned} \Delta(r(n)\chi(n)) &= -\Delta(r(n)\phi(n)) \sum_{j=0}^{n-1} \frac{w(j)\theta(j)f(j)}{W[\phi,\theta](j)} + \Delta(r(n)\theta(n)) \sum_{j=0}^{n-1} \frac{w(j)\phi(j)f(j)}{W[\phi,\theta](j)} \\ &\quad - \frac{r(n+1)\left(\phi(n+1)\theta(n) - \phi(n)\theta(n+1)\right) w(n)f(n)}{W[\phi,\theta](n)}, \end{aligned}$$

$$r(n)\nabla\chi(n) = -r(n)\nabla\phi(n)\sum_{j=0}^{n-1}\frac{w(j)\theta(j)f(j)}{W[\phi,\theta](j)} + r(n)\nabla\theta(n)\sum_{j=0}^{n-1}\frac{w(j)\phi(j)f(j)}{W[\phi,\theta](j)}.$$

Inserting $\chi(n)$ and the above three relations into (2.11) yields that χ satisfies (2.11) for $n \in [1,\infty)$. Hence, χ is a solution of (2.11). Since $\phi(n)$ and $\theta(n)$ are the solutions of the homogeneous linear equation (1.1), y defined by (2.13) is a solution of (2.11). Furthermore, it is easy to verify that y satisfies the initial conditions (2.12) by using (2.7).

On the other hand, the uniqueness of the solution of the initial value problem (2.11) and (2.12) is easily concluded by referring to $p(n) \neq 0$ for $n \in [-1,\infty)$. Hence, the proof is complete.

We now consider two transformations for equation (1.1) or equation (1.1*), which are useful in the following sections. First, we transform equation (1.1) into an equivalent equation with the weighted function identical to 1.

PROPOSITION 2.3. *Equation (1.1) is in l.c.c. at $n=\infty$ if and only if the following equation:*

$$-\nabla(p_1(n)\Delta z(n)) + q_1(n)z(n) + i[\Delta(r_1(n)z(n)) + r_1(n)\nabla z(n)] = \lambda z(n),\ n \in [0,\infty) \qquad (2.15)$$

is in l.c.c. at $n=\infty$, where

$$\begin{aligned} z(n) &= w^{1/2}(n)y(n),\\ q_1(n) &= (p(n)+p(n-1)+q(n))\,w^{-1}(n) - p_1(n) - p_1(n-1),\\ p_1(n) &= p(n)\,(w(n+1)w(n))^{-1/2},\\ r_1(n) &= r(n)\,(w(n)w(n-1))^{-1/2}. \end{aligned}$$

PROOF. By Remark 2.1, (1.1) can be rewritten as

$$\begin{aligned} &-(p(n)-ir(n+1))\,y(n+1) - (p(n-1)+ir(n))\,y(n-1)\\ &\qquad + (q(n)+p(n)+p(n-1))\,y(n) = \lambda w(n)y(n), \qquad n \in [0,\infty). \end{aligned}$$

By setting $y(n) = w^{-1/2}(n)z(n)$ and multiplying two sides of the above equation by $w^{-1/2}(n)$, (2.15) can be directly derived from the above relation. On the other hand, $\sum_{n=0}^{\infty}|z(n)|^2 = \sum_{n=0}^{\infty} w(n)|y(n)|^2$, which implies that $z \in l^2[0,\infty)$ if and only if $y \in l^2_w[0,\infty)$. This completes the proof.

The following result is a direct consequence of Proposition 2.3 by using Remark 2.1.

COROLLARY 2.2. *Equation (1.1*) is in l.c.c. at $n=\infty$ if and only if the following equation:*

$$-h(n)y(n+1) + b(n)w^{-1}(n)y(n) - \bar{h}(n-1)y(n-1) = \lambda y(n), \qquad n \in [0,\infty) \qquad (2.16)$$

is in l.c.c. at $n=\infty$, where $h(-1) = c(-1)w^{-1/2}(0)$ and $h(n) = c(n)w^{-1/2}(n+1)w^{-1/2}(n)$, $n \geq 0$.

Second, we transform equation (1.1*) into an equivalent equation with the first leading coefficient identical to 1.

PROPOSITION 2.4. *Equation (1.1*) is in l.c.c. at $n=\infty$ if and only if the following equation:*

$$-z(n+1) + b(n)|g(n)|^2 z(n) - z(n-1) = \lambda w(n)|g(n)|^2 z(n), \qquad n \in [0,\infty), \qquad (2.17)$$

is in l.c.c. at $n=\infty$, where $g(n)$ is defined by $g(-1) := 1$, $g(0) := c^{-1}(-1)$, and

$$g(n) := \begin{cases} \dfrac{\bar{c}(-1)\bar{c}(1)\cdots\bar{c}(n-2)}{c(0)c(2)\cdots c(n-1)}, & n = 1,3,5,\dots,\\[2ex] \dfrac{\bar{c}(0)\bar{c}(2)\cdots\bar{c}(n-2)}{c(-1)c(1)\cdots c(n-1)}, & n = 2,4,\dots. \end{cases} \qquad (2.18)$$

PROOF. Let $y(n) = g(n)z(n)$. Multiplying (1.1*) by $\bar{g}(n)$, we get

$$\begin{aligned} &-c(n)g(n+1)\bar{g}(n)z(n+1) + b(n)|g(n)|^2 z(n) \\ &-\bar{c}(n-1)\bar{g}(n)g(n-1)z(n-1) = \lambda w(n)|g(n)|^2 z(n), \qquad n \in [0,\infty). \end{aligned} \tag{2.19}$$

From the definition of g, it follows that

$$c(n)g(n+1)\bar{g}(n) = 1, \qquad n \geq 0, \tag{2.20}$$

which together with (2.19) implies (2.17). In addition,

$$\sum_{n=0}^{\infty} w(n)|g(n)|^2|z(n)|^2 = \sum_{n=0}^{\infty} w(n)|y(n)|^2.$$

Hence, equation (1.1*) is in l.c.c. at $n = \infty$ if and only if equation (2.17) is in l.c.c. at $n = \infty$. This completes the proof.

Recently, Chen and Shi found the invariance of the limit-point and limit-circle cases for a class of second-order linear difference equations with real coefficients under a certain bounded perturbation [13, Lemma 2.4]. This result still holds for second-order linear difference equations with complex coefficients.

THEOREM 2.5. *Let $w(n) \equiv 1$ for $n \in [0,\infty)$ and $\{d(n)\}_{n=0}^{\infty}$ be a bounded real sequence. equation (1.1*) is in l.c.c. at $n = \infty$ if and only if the equation,*

$$-c(n)y(n+1) + (b(n) + d(n))y(n) - \bar{c}(n-1)y(n-1) = \lambda y(n), \qquad n \in [0,\infty), \tag{2.21}$$

is in l.c.c. at $n = \infty$.

PROOF. First consider the necessary. Suppose that (1.1*) is in l.c.c. at $n = \infty$. It suffices to show that each solution of the equation,

$$-c(n)y(n+1) + [b(n) + d(n)]y(n) - \bar{c}(n-1)y(n-1) = 0, \qquad n \in [0,\infty), \tag{2.22}$$

is in $l^2(0,\infty)$ by Theorem 2.3. equation (2.22) can be rewritten as

$$-c(n)y(n+1) + b(n)y(n) - \bar{c}(n-1)y(n-1) = -d(n)y(n). \tag{2.23}$$

By Remark 2.1 and Theorem 2.4, the general solution of (2.23) can be expressed as

$$y(n) = \alpha\varphi(n) + \beta\psi(n) - \sum_{j=0}^{n-1} \frac{(\varphi(j)\psi(n) - \varphi(n)\psi(j))\, d(j)y(j)}{c(j)\,(\varphi(j+1)\psi(j) - \varphi(j+1)\psi(j))} \tag{2.24}$$

for $n \in [-1,\infty)$, where α and β are any two constants. Since $\{d(n)\}_{n=0}^{\infty}$ is bounded, there is a positive constant K_1 such that $|d(n)| \leq K_1$ for all $n \in [0,\infty)$. It follows from (2.24) and (2.10) that for $n \geq -1$,

$$|y(n)| \leq |\alpha||\varphi(n)| + |\beta||\psi(n)| + K_1 \sum_{j=0}^{n-1} (|\varphi(j)||\psi(n)| + |\varphi(n)||\psi(j)|)\,|y(j)|. \tag{2.25}$$

Let

$$y_1(n) = \frac{|y(n)|}{|\varphi(n)| + |\psi(n)|}, \qquad n \geq 0.$$

It follows from (2.25) that

$$y_1(n) \le |\alpha| + |\beta| + 2K_1 \sum_{j=0}^{n-1} \left(|\varphi(j)|^2 + |\psi(j)|^2\right) y_1(j), \qquad n \ge 0.$$

By [13, Lemma 2.3], we have

$$y_1(n) \le (|\alpha| + |\beta|) \exp\left(2K_1 \sum_{j=0}^{n-1} \left(|\varphi(j)|^2 + |\psi(j)|^2\right)\right),$$

for $n \ge 0$. By the assumption, $\varphi, \psi \in l^2(0,\infty)$. So, there exists a positive constant K_2 such that $y_1(n) \le K_2$ for all $n \in [0,\infty)$. Then,

$$|y(n)| \le K_2 \left(|\varphi(n)| + |\psi(n)|\right), \qquad n \ge 0,$$

which implies that $y \in l^2(0,\infty)$. Therefore, (2.21) is in l.c.c. at $n = \infty$.

Finally, consider the sufficiency. Suppose that (2.21) is in l.c.c. at $n = \infty$. equation (1.1*) can be regarded as a perturbation of (2.21); that is, equation (1.1*) can be rewritten as

$$-c(n)y(n+1) + \left(b(n) + d(n) - d(n)\right) y(n) - \bar{c}(n-1)y(n-1) = \lambda y(n), \qquad n \in [0,\infty).$$

So, the sufficiency directly follows from the conclusion of the first part. The proof is complete.

3. SEVERAL CRITERIA OF THE LIMIT POINT CASE

In this section, we shall establish several criteria of the limit-point case for equation (1.1) or its alternate form (1.1*).

THEOREM 3.1. *Equation (1.1*) is in l.p.c. at $n = \infty$ if*

$$\sum_{n=0}^{\infty} \frac{(w(n+1)w(n))^{1/2}}{|c(n)|} = \infty. \tag{3.1}$$

PROOF. Assume the contrary. Suppose that (1.1*) is in l.c.c. at $n = \infty$. Then, φ and ψ, defined as in Section 2, are linearly independent solutions in $l^2_w(0,\infty)$ of (1.1*) by Proposition 2.2 and Theorem 2.3. It follows from (2.10) that

$$|c(n)(\varphi(n+1)\psi(n) - \varphi(n)\psi(n+1))| = 1, \qquad n \in [-1,\infty),$$

which implies that

$$\begin{gathered} |(w^{1/2}(n+1)\varphi(n+1))(w^{1/2}(n)\psi(n))| \\ +|(w^{1/2}(n)\varphi(n))(w^{1/2}(n+1)\psi(n+1))| \ge \frac{(w(n+1)w(n))^{1/2}}{|c(n)|}. \end{gathered} \tag{3.2}$$

By Cauchy's inequality, the left-hand side of (3.2) is summable and consequently, the right-hand side of (3.2) is summable. This is contrary to (3.1). Therefore, (1.1*) is in l.p.c. at $n = \infty$. This completes the proof.

The following result is a consequence of Theorem 3.1 by Remark 2.1.

COROLLARY 3.1. *Equation (1.1) is in l.p.c. at $n = \infty$ if*

$$\sum_{n=0}^{\infty} \frac{(w(n+1)w(n))^{1/2}}{\sqrt{p^2(n) + r^2(n+1)}} = \infty. \tag{3.3}$$

THEOREM 3.2. *Let $\{b(n)/w(n)\}_{n=0}^{\infty}$ be bounded. Then, equation (1.1*) is in l.p.c. at $n = \infty$ if and only if*

$$\sum_{n=0}^{\infty} w(n)|g(n)|^2 = \infty,$$

where $g(n)$ is defined by (2.18).

PROOF. By Corollary 2.2, (1.1*) is in l.p.c. at $n = \infty$ if and only if (2.16) is in l.p.c. at $n = \infty$. By using the boundedness of $\{b(n)/w(n)\}_{n=0}^{\infty}$ and by Theorem 2.5, (2.16) is in l.p.c. at $n = \infty$ if and only if

$$-h(n)y(n+1) - \bar{h}(n-1)y(n-1) = \lambda y(n), \qquad n \in [0,\infty), \tag{3.4}$$

is in l.p.c. at $n = \infty$, where $h(n)$ is defined as in Corollary 2.2. On the other hand, by Proposition 2.4, (3.4) is in l.p.c. at $n = \infty$ if and only if

$$-y(n+1) - y(n-1) = \lambda |G(n)|^2 y(n), \qquad n \in [0,\infty), \tag{3.5}$$

is in l.p.c. at $n = \infty$, where $G(n)$ is defined as in (2.18) by replacing $c(j)$ with $h(j)$ for $j \geq -1$, that is, $G(-1) = 1$, $G(0) = h^{-1}(-1)$, and

$$G(n) = \begin{cases} \dfrac{\bar{h}(-1)\bar{h}(1)\cdots\bar{h}(n-2)}{h(0)h(2)\cdots h(n-1)}, & n = 1,3,\dots, \\[2ex] \dfrac{\bar{h}(0)\bar{h}(2)\cdots\bar{h}(n-2)}{h(-1)h(1)\cdots h(n-1)}, & n = 2,4,\dots. \end{cases}$$

By Theorem 2.3, equation (3.5) is in l.p.c. if and only if the following equation,

$$-y(n+1) - y(n-1) = 0, \qquad n \in [0,\infty), \tag{3.6}$$

has at least one solution not in $l^2_{|G|^2}(0,\infty)$. It is seen that any solution of (3.6) satisfies

$$|y(2n-1)| = |y(-1)|, \quad |y(2n)| = |y(0)|, \qquad n \geq 1.$$

Then, equation (3.6) has at least one solution not in $l^2_{|G|^2}(0,\infty)$ if and only if $\sum_{n=0}^{\infty} |G(n)|^2 = \infty$. Further, from the definitions of h (see Corollary 2.2) and g (see (2.18)), we have

$$\sum_{n=0}^{\infty} |G(n)|^2 = \sum_{n=0}^{\infty} w(n)|g(n)|^2.$$

Therefore, equation (3.6) has at least one solution not in $l^2_{|G|^2}(0,\infty)$, i.e., equation (1.1*) is in l.p.c. at $n = \infty$ if and only if $\sum_{n=0}^{\infty} w(n)|g(n)|^2 = \infty$. This completes the proof.

THEOREM 3.3. *Let $w(n) \equiv 1$ for $n \geq 0$ and $p(n) > 0$ for $n \geq -1$. If there exist a positive integer N, a sequence of positive numbers $\{M(n)\}_{n=N}^{\infty}$, and three positive constants k_1, k_2, and k_3 such that for all $n \geq N$,*

1) $|r(n+1)| + |r(n)| \leq k_1 M(n)$,
2) $q(n) \geq -k_2 M(n)$,
3)
$$\frac{p^{1/2}(n-1)|\nabla M(n)|}{M^{1/2}(n)M(n-1)} \leq k_3,$$
4)
$$\sum_{n=N}^{\infty} \frac{1}{(p^2(n-1) + r^2(n))^{1/4} M^{1/2}(n)} = \infty,$$

then equation (1.1) is in l.p.c. at $n = \infty$.

PROOF. Suppose that $ly(n) = 0$ has a nonzero solution $z \in l^2(0,\infty)$. Then, $\sum_{n=0}^{\infty} |z(n)|^2 \le k$ for some positive constant k and for $n \ge 0$,

$$\begin{aligned} &-\nabla(p(n)\Delta z(n))\frac{\bar{z}(n)}{M(n)} + q(n)z(n)\frac{\bar{z}(n)}{M(n)} \\ &+i[\Delta(r(n)z(n)) + r(n)\nabla z(n)]\frac{\bar{z}(n)}{M(n)} = 0. \end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned} \nabla\left(\frac{p(n)(\Delta z(n))\bar{z}(n)}{M(n)}\right) &= \frac{p(n-1)|\nabla z(n)|^2}{M(n)} - \frac{p(n-1)(\nabla M(n))(\nabla z(n))\bar{z}(n-1)}{M(n)M(n-1)} \\ &+\frac{q(n)|z(n)|^2}{M(n)} + \frac{ir(n+1)z(n+1)\bar{z}(n)}{M(n)} - \frac{ir(n)\bar{z}(n)z(n-1)}{M(n)}. \end{aligned} \tag{3.8}$$

Summing up (3.8) from N to n yields

$$\begin{aligned} \frac{p(n)(\Delta z(n))\bar{z}(n)}{M(n)} &= H(n) - \sum_{j=N}^{n} \frac{p(j-1)(\nabla M(j))(\nabla z(j))\bar{z}(j-1)}{M(j)M(j-1)} \\ &+\sum_{j=N}^{n} \frac{q(j)|z(j)|^2}{M(j)} + i\sum_{j=N}^{n} \frac{r(j+1)z(j+1)\bar{z}(j)}{M(j)} - i\sum_{j=N}^{n} \frac{r(j)z(j-1)\bar{z}(j)}{M(j)} + c_0, \end{aligned} \tag{3.9}$$

where

$$H(n) = \sum_{j=N}^{n} \frac{p(j-1)|\nabla z(j)|^2}{M(j)} \qquad \text{and} \qquad c_0 = \frac{p(N-1)(\Delta z(N-1))\bar{z}(N-1)}{M(N-1)}.$$

It is evident that $H(n)$ is nonnegative and nondecreasing on $[N,\infty)$. By the assumptions and Cauchy's inequality, (3.9) implies that

$$\mathrm{Re}\left\{\frac{p(n)(\Delta z(n))\bar{z}(n)}{M(n)}\right\} \ge H(n) - k_3k^{1/2}H^{1/2}(n) - k_2k - 2k_1k - |c_0|. \tag{3.10}$$

Assume that $\lim_{n\to\infty} H(n) = \infty$. Then, there exists a positive integer $N_1 \ge N$ such that the right-hand side of (3.10) is positive for all $n \ge N_1$ and consequently,

$$\mathrm{Re}\,\{(\Delta z(n))\bar{z}(n)\} > 0, \qquad n \ge N_1, \tag{3.11}$$

that is,

$$\frac{1}{2}\left(z(n+1)\bar{z}(n) + \bar{z}(n+1)z(n)\right) - \bar{z}(n)z(n) > 0, \qquad n \ge N_1. \tag{3.12}$$

It is seen from (3.11) that $z(n) \ne 0$ for $n \ge N_1$. Then, (3.12) implies that

$$\mathrm{Re}\left\{\frac{z(n+1)}{z(n)}\right\} > 1, \qquad n \ge N_1, \tag{3.13}$$

and then

$$\left|\frac{z(n+1)}{z(n)}\right| \ge 1, \qquad n \in [N_1,\infty).$$

So, $z \notin l^2(0,\infty)$, which is contrary to the assumption $z \in l^2(0,\infty)$. Hence, $\lim_{n\to\infty} H(n) < \infty$ by referring to the fact that $\{H(n)\}_{n=N}^{\infty}$ is nondecreasing. Suppose that all the solutions of $lz(n) = 0$

are in $l^2(0,\infty)$. By Proposition 2.2, ϕ and θ are linearly independent solutions of $lz(n)=0$ and are both in $l^2(0,\infty)$. From (2.9), we have

$$\begin{aligned}\frac{(p^2(n-1)+r^2(n))^{1/4}}{M^{1/2}(n)}&(|\phi(n)\nabla\theta(n)|+|(\nabla\phi(n))\theta(n)|)\\ &\geq \frac{1}{(p^2(n-1)+r^2(n))^{1/4}M^{1/2}(n)}.\end{aligned}\tag{3.14}$$

However, by using Cauchy's inequality and the inequality $\lim_{n\to\infty}H(n)<\infty$ for ϕ and θ, and by assumption 1), we have

$$\begin{aligned}&\sum_{j=N}^{\infty}\frac{(p^2(j-1)+r^2(j))^{1/4}}{M^{1/2}(j)}(|\phi(j)\nabla\theta(j)|+|(\nabla\phi(j))\theta(j)|)\\ &\leq \sum_{j=N}^{\infty}\left(\frac{p^2(j-1)}{M^2(j)}+k_1^2\right)^{1/4}(|\phi(j)\nabla\theta(j)|+|(\nabla\phi(j))\theta(j)|)\\ &\leq\left\{\sum_{j=N}^{\infty}\left(\frac{p^2(j-1)}{M^2(j)}+k_1^2\right)^{1/2}|\nabla\theta(j)|^2\right\}^{1/2}\left\{\sum_{j=N}^{\infty}|\phi(j)|^2\right\}^{1/2}\\ &\quad+\left\{\sum_{j=N}^{\infty}\left(\frac{p^2(j-1)}{M^2(j)}+k_1^2\right)^{1/2}|\nabla\phi(j)|^2\right\}^{1/2}\left\{\sum_{j=N}^{\infty}|\theta(j)|^2\right\}^{1/2}\\ &\leq\left\{\sum_{j=N}^{\infty}\left(\frac{p(j-1)}{M(j)}+k_1\right)|\nabla\theta(j)|^2\right\}^{1/2}\left\{\sum_{j=N}^{\infty}|\phi(j)|^2\right\}^{1/2}\\ &\quad+\left\{\sum_{j=N}^{\infty}\left(\frac{p(j-1)}{M(j)}+k_1\right)|\nabla\phi(j)|^2\right\}^{1/2}\left\{\sum_{j=N}^{\infty}|\theta(j)|^2\right\}^{1/2}\\ &<+\infty.\end{aligned}$$

This implies that the left-hand side of (3.14) is summable. But by assumption 4), the right-hand side of (3.14) is not summable, which is a contradiction. Then (1.1) is in l.p.c. at $n=\infty$. This completes the proof.

Several remarks on the results of this section are listed as follows.

REMARK 3.1. Theorems 3.1–3.3 extend the relevant results of [9,13,10] for equation (1.2) with only real coefficients to equation (1.1) with complex coefficients.

REMARK 3.2.

(1) Theorem 3.2 can not be included by Theorem 3.1. For example, let $c(-1)=c(0)=1$, $c(2m-1)=c(2m)=m^4+m^4i$ for $m\geq 1$, and $b(n)=w(n)\equiv 1$ for $n\geq 0$ in equation (1.1*). Then, it follows from (2.18) that $|g(2m-1)|=1$ for $m\geq 1$ and then

$$\sum_{n=0}^{\infty}w(n)|g(n)|^2=\infty,$$

which implies that (1.1*) is in l.p.c. at $n=\infty$ by Theorem 3.2. However, it is evident that Theorem 3.1 can not be applied to this example.

(2) Theorem 3.3 can not be included by Theorems 3.1 and 3.2. Consider the following example: $p(n)=n^2$ for $n\geq -1$; $q(n)>-K$ for some positive constant K, $r(n)\equiv 1$ and $w(n)\equiv 1$ for $n\geq 0$ in equation (1.1). By setting $M(n)\equiv 1$ for $n\geq 0$, it is easy to verify that all

the assumptions of Theorem 3.3 are satisfied. So, equation (1.1) is in l.p.c. at $n = \infty$ by Theorem 3.3. However, since

$$\sum_{j=0}^{\infty} \frac{1}{(p^2(n) + r^2(n+1))^{1/2}} = \sum_{j=0}^{\infty} \frac{1}{(n^4+1)^{1/2}} < \infty,$$

Theorem 3.1 can not be applied to this example. On the other hand, equation (1.1) can be rewritten as equation (1.1*) with $b(n) = q(n) + n^2 + (n-1)^2$. Then, for $n \geq 0$,

$$\frac{b(n)}{w(n)} = q(n) + n^2 + (n-1)^2,$$

which implies that the sequence $\{q(n)+n^2+(n-1)^2\}_{n=0}^{\infty}$ is unbounded. Thus, Theorem 3.2 cannot be applied to this example, too.

4. SEVERAL CRITERIA OF THE LIMIT CIRCLE CASE

In this section, we establish several criteria of the limit-circle case for equation (1.1) or its alternate form (1.1*).

For convenience, first introduce some notations. Let $x = (x_1, x_2)^\top$ and $A = (a_{ij})$ be a two-dimensional vector and a 2×2 matrix, respectively. Define their norms $\|x\|$ and $\|A\|$ as

$$\|x\| := |x_1| + |x_2|, \qquad \|A\| := \sum_{i=1}^{2} \sum_{j=1}^{2} |a_{ij}|.$$

They satisfy

$$\|Ax\| \leq \|A\| \|x\|.$$

THEOREM 4.1. *Let* $\sum_{n=0}^{\infty} w(n) < \infty$, $\sum_{n=0}^{\infty} |r(n)| < \infty$, $\sum_{n=0}^{\infty} \left(p^2(n) + r^2(n+1)\right)^{-1/2} < \infty$, *and* $\sum_{n=0}^{\infty} |q(n)| < \infty$. *Then, equation (1.1) is in l.c.c. at* $n = \infty$.

PROOF. By letting

$$z(n) = (-p(n) + ir(n+1))(y(n+1) - y(n)),$$

equation (1.1) can be transformed into the following system:

$$\Delta Y(n) = F(n) Y(n-1), \qquad n \geq 1, \tag{4.1}$$

where $Y(n) = (y(n), z(n))^\top$ and

$$F(n) = \begin{pmatrix} 0 & \dfrac{1}{-p(n-1) + ir(n)} \\ \lambda w(n) - q(n) + i(r(n) - r(n+1)) & \dfrac{\lambda w(n) - q(n) - i(r(n) + r(n+1))}{-p(n-1) + ir(n)} \end{pmatrix}. \tag{4.2}$$

Since all the entries of $F(n)$ are summable by the assumptions, $\sum_{j=0}^{\infty} \|F(j)\|$ converges. On the other hand, it follows from (4.1) that

$$\|Y(n)\| - \|Y(n-1)\| \leq \|F(n)\| \|Y(n-1)\|, \qquad n \geq 1,$$

which implies that there exists a positive constant K such that for $n \geq 0$

$$\begin{aligned} \|Y(n)\| &\leq (1 + \|F(n)\|) \|Y(n-1)\| \\ &\leq (1 + \|F(n)\|)(1 + \|F(n-1)\|) \cdots (1 + \|F(1)\|) \|Y(0)\| \\ &\leq \exp\left(\sum_{j=1}^{n} \|F(j)\|\right) \|Y(0)\| \leq K. \end{aligned}$$

Hence, $|y(n)| \le K$ for $n \ge 0$ and for each solution $y(n)$ of equation (1.1) and consequently,

$$\sum_{n=0}^{\infty} w(n)|y(n)|^2 \le K^2 \sum_{n=0}^{\infty} w(n) < \infty.$$

So, equation (1.1) is in l.c.c. at $n = \infty$. This completes the proof.

Now consider equation (1.1*). Let $u(n)$ be any solution of the following equation:

$$-c(n)y(n+1) + b(n)y(n) - \bar{c}(n-1)y(n-1) = 0, \qquad n \in [0, \infty). \tag{4.3}$$

Then,

$$u(n+1) = -\frac{\bar{c}(n-1)}{c(n)} u(n-1) + \frac{b(n)}{c(n)} u(n), \qquad n \ge 0, \tag{4.4}$$

$$u(n) = -\frac{\bar{c}(n-2)}{c(n-1)} u(n-2) + \frac{b(n-1)}{c(n-1)} u(n-1), \qquad n \ge 1. \tag{4.5}$$

Set $t(n) = (\bar{c}(n-1))/c(n)$ and $s(n) = b(n)/c(n)$. Inserting (4.5) into (4.4) yields

$$u(n+1) = (s(n)s(n-1) - t(n))\, u(n-1) - s(n)t(n-1)u(n-2), \qquad n \ge 1. \tag{4.6}$$

It follows from (4.4)–(4.6) that

$$\begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = S(0) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$

and

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = S(n) \begin{pmatrix} u(n-1) \\ u(n-2) \end{pmatrix}, \qquad n \ge 1,$$

where

$$S(0) = \begin{pmatrix} s(0) & -t(0) \\ 1 & 0 \end{pmatrix},$$
$$S(n) = \begin{pmatrix} s(n)s(n-1) - t(n) & -s(n)t(n-1) \\ s(n-1) & -t(n-1) \end{pmatrix}, \qquad n \ge 1.$$

Let

$$W(n) = \operatorname{diag}\{w(n+1, w(n)\},$$
$$T(2n) = S(2n)S(2n-2)\cdots S(2)S(0),$$
$$T(2n+1) = S(2n+1)S(2n-1)\cdots S(3)S(1), \qquad n \ge 0,$$

and

$$D(n) = \sum_{j=0}^{n} T^*(j)W(j)T(j), \qquad n \ge 0.$$

Clearly, $S(n)$ is nonsingular by referring to $c(n) \ne 0$ for $n \ge -1$. Then, $D(n)$ is a positive matrix.

Theorem 4.2. *Equation (1.1*) is in l.c.c. at $n = \infty$ if and only if $\{\|D(n)\|\}_{n=0}^{\infty}$ is bounded.*

Proof. By Theorem 2.3, it suffices to show that all the solutions of (4.3) is in $l_w^2(0, \infty)$ if and only if $\{\|D(n)\|\}_{n=0}^{\infty}$ is bounded.

First consider the sufficiency. Suppose that $\{\|D(n)\|\}_{n=0}^{\infty}$ is bounded. Let $u(n)$ be any solution of (4.3). Then,

$$\begin{aligned}
&\sum_{j=0}^{n}\left(w\,(j+1)\,|u(j+1)|^2 + w\,(j)\,|u\,(j)\,|^2\right)\\
&\quad= \sum_{j=0}^{n}(\bar{u}(j+1),\ \bar{u}(j))W(j)\begin{pmatrix} u(j+1)\\ u(j)\end{pmatrix}\\
&\quad= \sum_{j=0}^{n}(\bar{u}(j-1),\ \bar{u}(j-2))S^*(j)W(j)S(j)\begin{pmatrix} u(j-1)\\ u(j-2)\end{pmatrix}\\
&\quad= (\bar{u}(0),\ \bar{u}(-1))\sum_{j=0}^{n}(T^*(j)W(j)T(j))\begin{pmatrix} u(0)\\ u(-1)\end{pmatrix}\\
&\quad= (\bar{u}(0),\ \bar{u}(-1))D(n)\begin{pmatrix} u(0)\\ u(-1)\end{pmatrix}.
\end{aligned}\tag{4.7}$$

Since $\{\|D(n)\|\}_{n=0}^{\infty}$ is bounded, there exists a constant $K_1 > 0$ such that

$$\sum_{j=0}^{\infty}\left(w(j+1)|u(j+1)|^2 + w(j)|u(j)|^2\right) \le K_1\left(|u(-1)|^2 + |u(0)|^2\right) < \infty.$$

Therefore, all the solutions of (4.3) are in $l_w^2(0,\infty)$.

Finally, consider the necessary. Suppose that all the solutions of (4.3) are in $l_w^2(0,\infty)$. Let $z(n)$ be any solution of (4.3). Then $z \in l_w^2(0,\infty)$ and

$$\sum_{j=0}^{n}\left(w(j+1)|z(j+1)|^2 + w(j)|z(j)|^2\right) \le \sum_{j=0}^{\infty}\left(w(j+1)|z(j+1)|^2 + w(j)|z(j)|^2\right) < \infty. \tag{4.8}$$

It follows from (4.7) and (4.8) that for any fixed $z(-1), z(0) \in \mathbf{C}$, there exists a constant K_2 such that

$$(\bar{z}(0),\ \bar{z}(-1))D(n)\begin{pmatrix} z(0)\\ z(-1)\end{pmatrix} \le K_2, \qquad n \ge 0. \tag{4.9}$$

Hence, $\{\|D(n)\|\}_{n=0}^{\infty}$ is bounded by using the fact that $D(n)$ is positive. This completes the proof.

REMARK 4.1. Theorem 4.2 extends Theorem 3.1 of [13].

We now establish some sufficient conditions of the limit-point and limit-circle cases for equation (1.1*) in terms of the eigenvalues of $S^*(n)S(n)$. Since $S(n)$ is nonsingular, $S^*(n)S(n)$ is positive. Further, we have

$$S^*(n)S(n) = \begin{pmatrix} |s(n)s(n-1) - t(n)|^2 + |s(n-1)|^2 & \zeta(n)\\ \bar{\zeta}(n) & (|s(n)|^2+1)|t(n-1)|^2\end{pmatrix}, \tag{4.10}$$

where

$$\zeta(n) = -\left((|s(n)|^2+1)\bar{s}(n-1) - s(n)\bar{t}(n)\right)t(n-1).$$

Let $\mu(n)$ and $\lambda(n)$ be the eigenvalues of $S^*(n)S(n)$ with $\mu(n) \le \lambda(n)$. Then, $\mu(n)$ and $\lambda(n)$ are positive and

$$\mu(n)I_2 \le S^*(n)S(n) \le \lambda(n)I_2, \qquad n \ge 0.$$

THEOREM 4.3. *Let $w(n) \equiv 1$.*

(1) *If $\liminf_{n\to\infty}\mu(n) > 1$, then equation (1.1*) is in l.p.c. at $n = \infty$.*
(2) *If $\limsup_{n\to\infty}\lambda(n) < 1$, then equation (1.1*) is in l.c.c. at $n = \infty$.*

PROOF. Since the proof is similar to that of [13, Theorem 3.3], the details are omitted.

COROLLARY 4.1. *Let* $w(n) \equiv 1$. *If*

$$\limsup_{n\to\infty} \left(|s(n)s(n-1)-t(n)|^2 + |s(n-1)|^2 + (|s(n)|^2+1)|t(n-1)|^2\right) < 1,$$

then equation (1.1) is in l.c.c. at* $n = \infty$.

PROOF. It follows from (4.10) that for $n \geq 0$, the characteristic equation for $S^*(n)S(n)$ is

$$\det\left(\lambda I - S^*(n)S(n)\right) = \lambda^2 - \tau(n)\lambda + \gamma(n) = 0, \tag{4.11}$$

where

$$\tau(n) = |s(n)s(n-1)-t(n)|^2 + |s(n-1)|^2 + \left(|s(n)|^2+1\right)|t(n-1)|^2,$$
$$\gamma(n) = \left(|s(n)s(n-1)-t(n)|^2 + |s(n-1)|^2\right)\left(|s(n)|^2+1\right)|t(n-1)|^2 - |\zeta(n)|^2.$$

Since $\mu(n)$ and $\lambda(n)$ are the two eigenvalues of $S^*(n)S(n)$ with $\mu(n) \leq \lambda(n)$, they are the two roots of (4.11). So, $\mu(n) + \lambda(n) = \tau(n)$ by the assumption, which implies that $\lambda(n) < \tau(n)$ by referring to the fact that $\mu(n)$ and $\lambda(n)$ are positive. So, it follows from the assumption that

$$\limsup_{n\to\infty} \lambda(n) \leq \limsup_{n\to\infty} \tau(n) < 1.$$

Hence, equation (1.1*) is in l.c.c. at $n = \infty$ by Theorem 4.3. This completes the proof.

EXAMPLE 4.1. Consider equation (1.1*) with $w(n) \equiv 1$, $c(n) = -4^n + 4^n i$, and $b(n) = 4^n$. Clearly,

$$|s(n)s(n-1)-t(n)|^2 + |s(n-1)|^2 + (|s(n)|^2+1)|t(n-1)|^2 = \frac{21}{32} < 1.$$

Therefore, equation (1.1*) is in l.c.c. at $n = \infty$ by Corollary 4.1.

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